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Boundary States of D-branes in AdS_3 Based on Discrete Series

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Abstract

We study the D-branes in the Lorentzian AdS_3 background from the viewpoints of boundary states, emphasizing the role of open-closed duality in string theory. Working on the world-sheet with the Lorentzian signature, we construct the Cardy states based on the discrete series. We show that they are compatible with (1) the unitarity and normalizability, and (2) the spectral flow symmetry, in the open string spectrum. We also discuss the brane interpretation of them.

We further show that, in the case of superstring on $AdS_3 \times S^3 \times T^4$, our Cardy states yield an infinite number of physical BPS states in the open string channel, on which the spectral flows consistently act.

1 Introduction

Closed string theory on AdS_3 background has been a subject attracting much attentions of string theorists due to several reasons. Among other things it provides a non-trivial example of solvable string theories on a non-compact curved space-time, and was studied from this motivation in the pioneering works [1]. More recently it has been studied from the viewpoints of the AdS/CFT correspondence [2] at the stringy level in the papers [3, 4, 5] and also in the huge number of subsequent works (a detailed reference list is presented in e.g. [6], and recent works on this subject have been given in [7]).

However, we only have comparably few works about the *open* string sectors of AdS_3 string theory, in other words, the aspects of D-branes in AdS_3 background [8, 9, 10, 11, 12, 13, 14].

The main purpose of this paper is to propose a boundary state description of D-branes in (super) string theory on AdS_3 background, which is known to be described by $SL(2;\mathbf{R})$ WZW model. It is well-known to construct the complete basis of boundary states in general WZW model, called "Ishibashi states" [15]. Thus it is easy to solve the general gluing conditions on any representation space of the current algebra. However, the brane interpretation of boundary states in the $SL(2;\mathbf{R})$ WZW model is still a difficult problem. Main difficulty is originating from the fact that we have infinitely many representations in the physical Hilbert space and thus infinitely many Ishibashi states, which sharply contrasts with the feature of $SU(2)$ WZW model. An important constraint to determine the spectrum of allowed boundary states is "Cardy condition" [16], which embodies the open-closed string duality. In addition to the Cardy condition we shall take a criterion: *any states appearing in both open and closed string channels of the cylinder amplitudes should be consistent with the requirement of no-ghost theorem and normalizability*, which is our starting point of discussion. We shall further assume the consistency with the actions of spectral flows, which is natural if comparing with the analysis of classical solutions presented in [11, 13].

This paper is organized as follows; in section 2, we shortly review some familiar results about the D-branes in AdS_3 background. In section 3, which is the main section of this paper, we study the boundary states based on the discrete series and discuss their brane interpretation. We further argue the open string spectrum of the on-shell BPS states. Section 4 is devoted to the discussions about several open problems.

2 D-branes in AdS_3 Space

In this section we present a short review on the known results of D-branes in AdS_3 space. AdS_3 space can be defined as a hyper surface in 4 dimensional flat space with the signature (2,2) such as

$$(X_0)^2 - (X_1)^2 - (X_2)^2 + (X_3)^2 = l_{AdS}^2 \ , \tag{2.1}$$

where l_{AdS} is the radius of AdS_3 space. From now on we set $l_{AdS} = 1$. This space is the $SL(2;\mathbf{R})$ group manifold and a useful parametrization can be taken as

$$g = \begin{pmatrix} X_0 + X_1 & -X_2 + X_3 \\ -X_2 - X_3 & X_0 - X_1 \end{pmatrix} \ . \tag{2.2}$$

Another useful parametrization is given by

$$g = e^{i\sigma_2 \frac{t+\theta}{2}} e^{\sigma_3 \rho} e^{i\sigma_2 \frac{t-\theta}{2}} \ , \tag{2.3}$$

or equivalently,

$$\begin{aligned} X_0 &= \cos t \cosh \rho \ , \quad X_1 = \cos\theta \sinh \rho \ , \\ X_2 &= \sin\theta \sinh\rho \ , \quad X_3 = \sin t \cosh \rho \ , \end{aligned} \tag{2.4}$$

which are called the "global coordinates" and the AdS_3 metric is written under this parametrization as

$$ds^2 = d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho d\theta^2 \ . \tag{2.5}$$

String theory in the group manifold can be described by the WZW model. We shall now work on the $SL(2;\mathbf{R})_{k+2}$ WZW model (the reason why we consider the level $k+2$ rather than the level k is nothing but the later convenience) of which action is given by

$$S = \frac{i(k+2)}{4\pi} \int d^2 z \mathrm{Tr}(g^{-1}\partial g \cdot g^{-1}\tilde{\partial} g) + \frac{i(k+2)}{12\pi} \int_{\mathcal{B}} \mathrm{Tr}(g^{-1}dg)^{\wedge 3} \ , \tag{2.6}$$

where $\mathcal{B}$ is the manifold whose boundary is the world-sheet. Since we will later work on the world-sheet with Lorentzian signature, we use the light-cone coordinates $z = e^{i(\tau+\sigma)}$, $\tilde{z} = e^{i(\tau-\sigma)}$. This theory has the left-moving and right-moving conserved currents

$$j(z) = -\frac{k+2}{2} \partial g \cdot g^{-1} = j^a T^b \eta_{ab} \ , \quad \tilde{j}(\tilde{z}) = \frac{k+2}{2} g^{-1} \tilde{\partial} g = \tilde{j}^a T^b \eta_{ab} \ , \tag{2.7}$$

where we used the following basis[1]

$$T^3 = \frac{1}{2}\sigma^2, \quad T^{\pm} = \frac{i}{2}(\sigma^3 \pm i\sigma^1) \ , \tag{2.8}$$

[1]In the present convention the space-time energy operator associated to the global coordinate t is $j_0^3 - \tilde{j}_0^3$.

which satisfy the commutation relations

$$[T^3,\ T^{\pm}] = \pm T^{\pm}\ , \quad [T^+,\ T^-] = -2T^3\ . \tag{2.9}$$

The metric on these basis is defined by $\eta^{ab} = -2\mathrm{Tr}(T^a T^b)$ (namely, the non-zero components of η^{ab} are $\eta^{33} = -1$, $\eta^{+-} = \eta^{-+} = 2$) and we set $\eta_{ab} = (\eta^{ab})^{-1}$.

D-branes in WZW model are described by the world-sheet with boundary as established in many works [17]. In the open string description the gluing condition of the left and right moving currents is generally given by

$$j = \omega(\tilde{j})|_{z=\tilde{z}}\ , \tag{2.10}$$

where ω is an automorphism of $SL(2;\mathbf{R})$ Lie algebra. Alternatively, by exchanging the roles of world-sheet coordinates τ and σ, we may rewrite this condition in the closed string picture

$$(j^a_n + \omega(\tilde{j})^a_{-n})|B\rangle = 0\ , \tag{2.11}$$

where $|B\rangle$ is a boundary state which corresponds to a D-brane and the world-volume of such boundary state can be identified with the (twined) conjugacy class [17]

$$\mathcal{C}^{\omega}(h) = \{hg\omega(h)^{-1},\ ^{\forall}h \in SL(2;\mathbf{R})\}\ . \tag{2.12}$$

If ω is an inner automorphism, we can set $\omega = \mathbf{1}$ by the rotation of the currents and the gluing condition (2.11) is reduced to the simplest form

$$(j^a_n + \tilde{j}^a_{-n})|B\rangle = 0\ . \tag{2.13}$$

We can parametrize the corresponding conjugacy classes as

$$\mathrm{Tr}g = 2X_0 = 2\tilde{C}\ , \tag{2.14}$$

where $\tilde{C}$ is a constant. In this parametrization the hyper surface (2.1) becomes

$$(X_3)^2 - (X_1)^2 - (X_2)^2 = 1 - \tilde{C}^2, \tag{2.15}$$

and this equation implies that the geometry of D-branes in $SL(2;\mathbf{R})$ group manifold is dS_2 space ($\tilde{C}^2 > 1$) or hyperbolic plane (H_2) ($\tilde{C}^2 < 1$) [8, 10]. There are also the cases of "degenerated D-branes" whose shapes are the light-cone ($\tilde{C}^2 = 1$) or the point ($g = 1$). It was discussed in [10] based on the classical analysis of DBI action that these D-branes are the unphysical branes with the supercritical electric fields.

On the other hand, if ω is an outer automorphism, we obtain a different geometry of brane. Let us choose ω as

$$\omega(h) = \omega h \omega^{-1} \ , \ \omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \tag{2.16}$$

which defines an outer automorphism, since ω does not belong to $SL(2;\mathbf{R})$. Then, the gluing condition becomes

$$\left\{ \begin{array}{rcl} (j^3_n - \tilde{j}^3_{-n})|B\rangle &=& 0 \\ (j^\pm_n - \tilde{j}^\mp_{-n})|B\rangle &=& 0 \ , \end{array} \right. \tag{2.17}$$

and the "twined conjugacy classes" can be expressed as

$$\begin{aligned} \mathcal{C}^\omega(h) &= \{hg(\omega h \omega^{-1})^{-1}, \ {}^\forall h \in SL(2;\mathbf{R})\} \\ &= \{hg\omega h^{-1} \cdot \omega^{-1}, \ {}^\forall h \in SL(2;\mathbf{R})\} \ . \end{aligned} \tag{2.18}$$

We can thus characterize these classes by a constant C as

$$\mathrm{Tr}(g\omega) = -2X_2 = 2C \ . \tag{2.19}$$

The geometry of such D-branes can be described as the hyper surface (2.1)

$$(X_0)^2 + (X_3)^2 - (X_1)^2 = 1 + C^2 \ , \tag{2.20}$$

which is an AdS_2 space. As was shown in [10], such AdS_2-branes are the physical D-branes with the subcritical electric field. We will concentrate on this case in this paper.

It is known that the $SL(2;\mathbf{R})$ WZW model possesses the symmetry called "spectral flow" $U_w \otimes \tilde{U}_{\tilde{w}}$ (see e.g. [18])

$$\begin{array}{l} \left\{ \begin{array}{rcl} U_w \, j^3(z) \, U_w^{-1} &=& j^3(z) + \dfrac{k+2}{2}\dfrac{w}{z} \\ U_w \, j^\pm(z) U_w^{-1} &=& z^{\mp w} j^\pm(z) \ , \end{array} \right. \\ \left\{ \begin{array}{rcl} \tilde{U}_{\tilde{w}} \, \tilde{j}^3(\tilde{z}) \, \tilde{U}_{\tilde{w}}^{-1} &=& \tilde{j}^3(\tilde{z}) + \dfrac{k+2}{2}\dfrac{\tilde{w}}{\tilde{z}} \\ \tilde{U}_{\tilde{w}} \, \tilde{j}^\pm(\tilde{z}) \, \tilde{U}_{\tilde{w}}^{-1} &=& \tilde{z}^{\mp \tilde{w}} \tilde{j}^\pm(\tilde{z}) \ , \end{array} \right. \end{array} \tag{2.21}$$

which are parametrized by integers w and $\tilde{w}$. In [19] it was claimed that the closed string Hilbert space in the case of the universal cover of $SL(2;\mathbf{R})$ should be extended by the spectral flow with $w = -\tilde{w}$ ("winding number")[2]. The similar claim has been addressed in [13] for the open string Hilbert space.

In our convention of currents the spectral flow $U_w \otimes \tilde{U}_{\tilde{w}}$ means the transformation

$$\begin{aligned} g(z,\tilde{z}) &\longmapsto z^{wT^3}\, g(z,\tilde{z})\, \tilde{z}^{-\tilde{w}T^3} \\ &\equiv e^{\frac{\tau+\sigma}{2} iw\sigma^2}\, g(z,\tilde{z})\, e^{-\frac{\tau-\sigma}{2} i\tilde{w}\sigma^2} \,. \end{aligned} \tag{2.22}$$

In this paper we shall work on the single cover of $SL(2;\mathbf{R})$, which corresponds to the Wick rotation of thermal AdS_3 space. This is because we are interested in the open-closed string duality, in which the roles of world-sheet coordinates τ and σ are exchanged, and hence we need to consider the winding sectors along not only the space-like circle but also the time-like one in some situations. In this case we can a priori choose the left-moving and right-moving windings w, $\tilde{w}$ independently. However, the consistency with the gluing condition brings us some constraints on the windings w, $\tilde{w}$. For the gluing condition (2.13), the allowed spectral flows should have the forms; $U_w \otimes \tilde{U}_{-w}$, in other words,

$$\begin{cases} t & \mapsto \quad t + w\tau \\ \theta & \mapsto \quad \theta + w\sigma \\ \rho & \mapsto \quad \rho \,, \end{cases} \tag{2.23}$$

in terms of the global coordinates (2.4) in the closed string channel.

On the other hand, the spectral flows allowed for (2.17) are given as $U_w \otimes \tilde{U}_w$,

$$\begin{cases} t & \mapsto \quad t + w\sigma \\ \theta & \mapsto \quad \theta + w\tau \\ \rho & \mapsto \quad \rho \,, \end{cases} \tag{2.24}$$

which generates the non-trivial winding sectors along the time-like circle.

3 Boundary States in String Theory on AdS_3

In this section we try to construct the consistent boundary states describing the D-branes in AdS_3 space. As we declared in the introduction, our main criterion is the open-closed

[2] The time direction is uncompactified in the universal cover of $SL(2;\mathbf{R})$ and w and $\tilde{w}$ must be related [19].

string duality with the requirement that the spectra in both open and closed string channels should be compatible with the unitarity and normalizability, and moreover the spectral flow symmetry. We shall here only treat the states belonging to the principal discrete series (short string sector[3]) and leave the case of the principal continuous series in future works.

To fix the specific background let us consider the superstring vacua $AdS_3 \times S^3 \times T^4$, which is the most familiar example (see, e.g. [3]), although we shall mainly focus on the bosonic sector. The AdS_3 sector is described by $SL(2;\mathbf{R})_{k+2}$ super WZW model ($k+2$ is the level of bosonic current) and the S^3 sector is described by $SU(2)_{k-2}$ super WZW model. We here assume $k \in \mathbf{Z}$ and $k > 2$.

In the string theory on Lorentzian AdS_3 it is known [19] that the physical Hilbert space should be constructed based on the representation spaces of discrete series with the constraints of no-ghost theorem [1, 20] and normalizability, and also based on the continuous series. We should suitably incorporate the degrees of freedom of spectral flow for both of these representations, as discussed in [19].

3.1 Open-closed Duality on Lorentzian World-sheet

Before presenting our main analysis we would like to make a few comments about the open-closed duality on the world-sheet with the *Lorentzian signature*, which we will use in the later discussions.

Firstly, in the standard argument on the Euclidean world-sheet, the open-closed duality for the cylinder amplitude is expressed as the following relation[4];

$$\int_0^\infty dT^{(c)}\, \eta(\tilde{\tau})^2\, Z_{\text{closed}}(\tilde{\tau}) = \int_0^\infty \frac{dT^{(o)}}{T^{(o)}}\, \eta(\tau)^2\, Z_{\text{open}}(\tau)\ , \tag{3.1}$$

where $\tau = iT^{(o)}$ denotes the open string modulus and $\tilde{\tau} \equiv -\dfrac{1}{\tau} = iT^{(c)}$ denotes the closed string modulus. The factors of η-functions are the contributions from bc-ghosts.

[3]In this paper we use the terminologies "short string" and "long string" according to [19]. Namely, the short string means the excitation corresponding to the discrete series (with arbitrary windings w) and the long string corresponds to the continuous series (with the non-zero windings). We call the sectors with non-zero w as the "winding strings" or "circular strings".

[4]Throughout this paper we denote the moduli for the open string channel by τ, z and those for the closed string channel by $\tilde{\tau}\left(\equiv -\dfrac{1}{\tau}\right)$, $\tilde{z}\left(\equiv \dfrac{z}{\tau}\right)$. We also write $\tilde{q} \equiv e^{2\pi i\tilde{\tau}}$, $\tilde{y} \equiv e^{2\pi i\tilde{z}}$, and so on.

Because of the simple identities

$$\int_0^\infty dT^{(c)} = \int_0^\infty \frac{dT^{(o)}}{T^{(o)\,2}} \ , \ \eta(\tilde{\tau})^2 = -i\tau\eta(\tau)^2 = T^{(o)}\eta(\tau)^2 \ , \tag{3.2}$$

we obtain

$$Z_{\text{closed}}(\tilde{\tau}) = Z_{\text{open}}(\tau) \ , \tag{3.3}$$

which is the standard statement of open-closed duality.

On the other hand, if working on the Lorentzian world-sheet, one must regard the moduli τ, $\tilde{\tau}$ $\left(\equiv -\frac{1}{\tau}\right)$ as real numbers. In this paper we take the convention $\tau = -T^{(o)}(<0)$, $\tilde{\tau} = T^{(c)}(>0)$ (such that $T^{(o)} = \frac{1}{T^{(c)}}$ holds as before). Then, (3.1) reduces to the following relation

$$Z_{\text{closed}}(\tilde{\tau}) = -iZ_{\text{open}}(\tau) \ , \tag{3.4}$$

instead of (3.3). Clearly the same formula holds also for the superstring cases.

3.2 Boundary States Based on Discrete Series

Let us work with the (bosonic) $SL(2;\mathbf{R})_{k+2}$ WZW model. Let $\hat{\mathcal{D}}_l^\pm$ be the representation space of discrete series ($+$: lowest weight representation, $-$: highest weight representation)[5]. We also express the representation transformed by the spectral flow as $\hat{\mathcal{D}}_l^{\pm\,(w)}$ as in [19]. The spectral flow parameter w ($=\tilde{w}$ for AdS_2-brane) was introduced as (2.21) in our convention. The unitarity and normalizability lead to the constraints $-1 < l < k-1$ for the l-quantum number [20, 19, 21]. Since we are here considering the single cover of AdS_3 space, the allowed values of l are $l = 0, 1, \ldots, k-2$. Quite interestingly, this range is equal to that for the integral representations of $SU(2)_{k-2}$.

[5]Under our convention the conformal weight of zero-mode states is given by $h = -\frac{l(l+2)}{4k}$, and the j_0^3-spectrum of the zero-mode states is $j_0^3 = \pm\left(\frac{l}{2}+n+1\right)$, ($n \in \mathbf{Z}_{\geq 0}$), for $\hat{\mathcal{D}}_l^\pm$ ($l > -2$). The double-sided representations ("degenerate representations") have the zero-mode spectra: $j_0^3 = \frac{l}{2}, \frac{l-2}{2}, \ldots, -\frac{l}{2}$ ($l \in \mathbf{Z}_{\geq 0}$), which are natural analogs of the unitary representations of $SU(2)$. On the other hand, the principal continuous series $\hat{\mathcal{C}}_{\lambda,\alpha}$ ($\lambda \in \mathbf{R}$, $0 \leq \alpha < 1$) corresponds to the branch $l = -1 + 2i\lambda$ (i.e. $h = \frac{1}{k}\left(\lambda^2 + \frac{1}{4}\right)$), and has the zero-mode spectrum: $j_0^3 = \alpha + n$, ($n \in \mathbf{Z}$).

The character with respect to the representation $\hat{\mathcal{D}}_l^{\pm\,(w)}$ is given by

$$\chi_l^{\pm\,(w)}(\tau,z) \equiv \mathrm{Tr}_{\hat{\mathcal{D}}_l^{\pm\,(w)}} q^{L_0-\frac{c}{24}} y^{j_0^3} = (-1)^w \frac{q^{-\frac{(l+1\mp kw)^2}{4k}} y^{\pm\frac{l+1\mp kw}{2}}}{-i\theta_1(\tau,\pm z)} , \tag{3.5}$$

where $q \equiv e^{2\pi i\tau}$, $y \equiv e^{2\pi i z}$.

Let us start with the Ishibashi state $|l,\pm\rangle_I$ based on the representation $\hat{\mathcal{D}}_l^{\pm}$ defined by the gluing condition (2.17) (with $a=0$) describing the AdS_2-branes. Let $\mathcal{B}(\hat{\mathcal{D}}_l^{\pm})$ be an orthonormal basis of $\hat{\mathcal{D}}_l^{\pm}$ composed of the eigenstates of j_0^3. We choose the phases of each bases $\mathbf{v} \in \mathcal{B}(\hat{\mathcal{D}}_l^{\pm})$ so that the matrix elements of all the currents j_n^3, $j_n^{\pm}$ are real numbers. Namely, we assume that $\langle \mathbf{v}_1 | j_n^3 | \mathbf{v}_2\rangle = \langle \mathbf{v}_2 | j_{-n}^3 | \mathbf{v}_1\rangle$, $\langle \mathbf{v}_1 | j_n^{\pm} | \mathbf{v}_2\rangle = \langle \mathbf{v}_2 | j_{-n}^{\mp} | \mathbf{v}_1\rangle$ for arbitrary $\mathbf{v}_1,\ \mathbf{v}_2 \in \mathcal{B}(\hat{\mathcal{D}}_l^{\pm})$. ($\mathcal{B}(\hat{\mathcal{D}}_l^{\pm})$ is not uniquely defined even under this requirement. However, such ambiguity is clearly harmless for our discussion.) We further denote the signature of norm $\langle \mathbf{v}|\mathbf{v}\rangle$ as $\epsilon_{\mathbf{v}}$ for each $\mathbf{v} \in \mathcal{B}(\hat{\mathcal{D}}_l^{\pm})$. (Recall that $\hat{\mathcal{D}}_l^{\pm}$ is not a unitary representation of affine algebra $\widehat{SL}(2;\mathbf{R})_{k+2}$.)

Under these preliminaries we can explicitly write down the Ishibashi state which satisfies the gluing condition (2.17)

$$|l,\pm\rangle_I = \sum_{\mathbf{v}\in\mathcal{B}(\hat{\mathcal{D}}_l^{\pm})} \epsilon_{\mathbf{v}} |\mathbf{v}\rangle \otimes \widetilde{|\mathbf{v}\rangle} . \tag{3.6}$$

The gluing condition (2.13) (for the conjugacy class with no twist) is also easily solved;

$$|l,\pm\rangle_I' = \sum_{\mathbf{v}\in\mathcal{B}(\hat{\mathcal{D}}_l^{\pm})} \epsilon_{\mathbf{v}} |\mathbf{v}\rangle \otimes T\widetilde{|\mathbf{v}\rangle} , \tag{3.7}$$

where T denotes the isomorphism $T\ :\ \hat{\mathcal{D}}_l^{\pm} \overset{\cong}{\to} \hat{\mathcal{D}}_l^{\mp}$ such that T maps the lowest (highest) weight vector in $\hat{\mathcal{D}}_l^{+}$ ($\hat{\mathcal{D}}_l^{-}$) to the highest (lowest) weight vector in $\hat{\mathcal{D}}_l^{-}$ ($\hat{\mathcal{D}}_l^{+}$), and satisfies

$$\begin{aligned} T\,\tilde{j}_n^3\,T &= -\tilde{j}_n^3 , \\ T\,\tilde{j}_n^{\pm}\,T &= -\tilde{j}_n^{\mp} . \end{aligned} \tag{3.8}$$

However, since the brane configuration described by (2.13) is known to be unphysical [10], we shall concentrate on the AdS_2-brane cases (2.17).

As we already pointed out, the spectral flow compatible with the gluing condition (2.17) is the type of (2.24), and we can similarly obtain the Ishibashi states for the flowed representations

$$|l,w,\pm\rangle_I = \sum_{\mathbf{v}\in\mathcal{B}(\hat{\mathcal{D}}_l^{\pm\,(w)})} \epsilon_{\mathbf{v}} |\mathbf{v}\rangle \otimes \widetilde{|\mathbf{v}\rangle} . \tag{3.9}$$

They are characterized by the following cylinder amplitudes

$$
{}_I\langle l,\pm,w|\tilde{q}^{H^{(c)}}\tilde{y}^{j_0^3}|l',\pm,w'\rangle_I=\delta_{ll'}\delta_{ww'}\,\chi_l^{\pm\,(w)}(\tilde{\tau},\tilde{z})\ , \tag{3.10}
$$

where $H^{(c)}=\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})$ and $\chi_l^{\pm\,(w)}(\tilde{\tau},\tilde{z})$ is defined in (3.5).

Now, we shall take the following boundary states which will be used as the building blocks of our Cardy states:

$$
\begin{aligned}
|l,w\rangle_I &= |l,w,+\rangle_I+|l,-w,-\rangle_I \\
&\equiv |l,w,+\rangle_I+|k-2-l,-w+1,+\rangle_I\ ,
\end{aligned} \tag{3.11}
$$

which are also used in [22] and we have

$$
\begin{aligned}
{}_I\langle l,w|\tilde{q}^{H^{(c)}}\tilde{y}^{j_0^3}|l',w'\rangle_I &= \delta_{ll'}\delta_{ww'}(-1)^w\left[\chi_l^{+\,(w)}(\tilde{\tau},\tilde{z})+\chi_l^{-\,(-w)}(\tilde{\tau},\tilde{z})\right] \\
&= \delta_{ll'}\delta_{ww'}(-1)^w\left[\frac{q^{-\frac{(l+1-kw)^2}{4k}}y^{\frac{l+1-kw}{2}}}{-i\theta_1(\tilde{\tau},\tilde{z})}+\frac{q^{-\frac{(l+1-kw)^2}{4k}}y^{-\frac{l+1-kw}{2}}}{i\theta_1(\tilde{\tau},\tilde{z})}\right] \\
&\equiv \delta_{ll'}\delta_{ww'}\,\chi_l^{(w)}(\tilde{\tau},\tilde{z})\ .
\end{aligned} \tag{3.12}
$$

The right hand side of the above identity (3.12) has a well-defined limit under $\tilde{z}\to 0$, such as

$$
\begin{aligned}
{}_I\langle l,w|\tilde{q}^{H^{(c)}}|l',w'\rangle_I &= \delta_{ll'}\delta_{ww'}(-1)^w\left[-(l+1-kw)\frac{\tilde{q}^{-\frac{(l+1-kw)^2}{4k}}}{\eta(\tilde{\tau})^3}\right] \\
&\equiv \delta_{ll'}\delta_{ww'}\,\chi_l^{(w)}(\tilde{\tau})\ .
\end{aligned} \tag{3.13}
$$

It may be interesting that the character $\chi_l^{(w=0)}(\tau,z)$ is formally equal to (the negative sign of) the character of the degenerate representation treated in [12].

It is worthwhile to notice that the summation of the characters $\sum\limits_{w\in 2\mathbf{Z}}\chi_l^{(w)}(\tau,z)$ has formally the quite reminiscent expression of the well-known character formula of $\widehat{SU}(2)_{k-2}$, as was pointed out in [18]. In fact[6],

$$
\chi_l^{SL(2)}(\tau,z)\equiv\sum_{w\in 2\mathbf{Z}}\chi_l^{(w)}(\tau,z)=\frac{\Theta_{-(l+1),-k}(\tau,z)-\Theta_{l+1,-k}(\tau,z)}{i\theta_1(\tau,z)}\ , \tag{3.14}
$$

[6] The range of summation $w\in 2\mathbf{Z}$ in $\chi_l^{SL(2)}(\tau,z)$ means that we sum all the windings w since we use the combination (3.11) as the building blocks.

and for $\widehat{SU}(2)_{k-2}$, the character is written as

$$\chi_l^{SU(2)}(\tau, z) = \frac{\Theta_{l+1,k}(\tau, z) - \Theta_{-(l+1),k}(\tau, z)}{i\theta_1(\tau, z)} \ . \tag{3.15}$$

This fact leads to a good modular property of the character $\chi_l^{SL(2)}$;

$$\chi_L^{SL(2)}(-\frac{1}{\tau}, \frac{z}{\tau}) = i \sum_{l=0}^{k-2} S_{Ll}^{(k-2)} \chi_l^{SL(2)}(\tau, z) \ , \tag{3.16}$$

where

$$S_{Ll}^{(k-2)} \equiv \sqrt{\frac{2}{k}} \sin\left(\pi \frac{(L+1)(l+1)}{k}\right) \tag{3.17}$$

is the well-known matrix of the modular transformation for $\widehat{SU}(2)_{k-2}$.

Of course, the power series of (3.14) is divergent for the usual range of modulus τ (i.e. $\mathrm{Im}\,\tau > 0$), since they include the negative level theta functions. Therefore, we must here take the Lorentzian signature on the world-sheet, namely, we must regard τ as a real number as in the calculation of partition function in the appendix of [19]. The extra factor i in the RHS of (3.16) reflects this fact and its existence matches the formula of open-closed duality in the Lorentzian world-sheet (3.4). We might still have a subtlety due to the fact that the power series does not absolutely converge. Such subtlety is likely to be unavoidable, if one intends to take account of the spectral flow, because the flowed representations have no lower and upper bounds of energy in general. Anyway, we shall proceed further assuming the validity of the formula (3.16).

Now, we would like to look for the suitable Cardy states based on the Ishibashi states (3.11). We shall start with the ansatz

$$|a\rangle_C = \sum_{l=0}^{k-2} \sum_{w \in 2\mathbf{Z}} \Psi_a(l, w) |l, w\rangle_I \ , \tag{3.18}$$

where the index a runs over the set of allowed Cardy states, which should be fixed later. It should be emphasized that the summation over the winding w is needed for good modular property. Because of the unitarity bound for the closed string spectrum, we must take the range of l-summation as $0 \le l \le k-2$.

We first notice the modular property of the character $\chi_l^{(w)}(\tau)$;

$$\sum_{w \in 2\mathbf{Z}} \sum_{l=0}^{k-2} \sqrt{\frac{2}{k}} \sin\left(\pi \frac{(L+1)(l+1-kw)}{k}\right) \chi_l^{(w)}(\tilde{\tau}) = -i \sum_{W \in 2\mathbf{Z}} \chi_L^{(W)}(\tau) \ , \tag{3.19}$$

by making use of the Poisson resummation formula. We can here assume $-1 < L < k-1$ without loss of generality because L appears only in the combination $L' = L+1-kW$ in the character $\chi_L^{(W)}(\tau)$. Moreover, since the time direction is compactified, the character in the open string channel $\chi_L^{(W)}(\tau)$ must again have the discrete quantum number L. So, we can assume $L = 0, 1, \ldots, k-2$, and the Cardy condition for the ansatz (3.18) can be written as

$$\Psi_a^*(l,w)\Psi_b(l,w) = \sum_{L=0}^{k-2} N_{ab}^{(o)}(L)\sqrt{\frac{2}{k}}\sin\left(\frac{\pi(L+1)(l+1)}{k}\right) , \tag{3.20}$$

where $N_{ab}^{(o)}(L) \in \mathbf{Z}_{\geq 0}$. This condition is formally the same as that of $\widehat{SU}(2)_{k-2}$ and the solution is well-known [16]. Assuming the diagonal modular invariant, the solution is given by

$$\begin{aligned} |L\rangle_C &= \sum_{l=0}^{k-2}\sum_{w\in 2\mathbf{Z}} \Psi_L(l,w)|l,w\rangle_I , \\ \Psi_L(l,w) &= \frac{S_{Ll}^{(k-2)}}{\sqrt{S_{0l}^{(k-2)}}} \\ &= \left(\frac{2}{k}\right)^{1/4} \frac{\sin\left(\frac{\pi(L+1)(l+1)}{k}\right)}{\sqrt{\sin\left(\frac{\pi(l+1)}{k}\right)}} . \end{aligned} \tag{3.21}$$

Using the Verlinde formula,

$$\frac{S_{L_1 l}^{(k-2)} S_{L_2 l}^{(k-2)}}{S_{0l}^{(k-2)}} = \sum_{L=0}^{k-2} N_{L_1,L_2}^{L} S_{Ll}^{(k-2)} , \tag{3.22}$$

where $N_{L_1 L_2}^L$ denotes the fusion matrix of $\widehat{SU}(2)_{k-2}$

$$N_{L_1 L_2}^L = \begin{cases} 1 & |L_1 - L_2| \leq L \leq \min(L_1 + L_2,\, 2(k-2) - L_1 - L_2) \text{ and } L \equiv |L_1 - L_2| \bmod 2 \\ 0 & \text{otherwise} , \end{cases} \tag{3.23}$$

we obtain as in the $SU(2)$ case,

$${}_C\langle L_1|\tilde{q}^{H^{(c)}}|L_2\rangle_C = -i\sum_{L=0}^{k-2} N_{L_1 L_2}^L \chi_L^{SL(2)}(\tau) , \tag{3.24}$$

where

$$\chi_L^{SL(2)}(\tau) \equiv \sum_{W\in 2\mathbf{Z}} \chi_L^{(W)}(\tau) \equiv \sum_{W\in 2\mathbf{Z}} \left[-(L+1-kW)\frac{q^{-\frac{(L+1-kW)^2}{4k}}}{\eta(\tau)^3} \right] , \tag{3.25}$$

as defined above. The identity (3.24) corresponds to the relation of the open-closed duality (3.4) we want.

As we already declared, our important criterion for the Cardy states is the requirement that the spectrum in both open and closed string channels should be consistent with the unitarity bound. Moreover, it is quite natural to require that the density of states in open string channel should be invariant under the spectral flow, as claimed in [13] based on the analysis of the classical open string solutions. The equation (3.24) actually possesses these properties, hence (3.21) is regarded as the desired solution of Cardy condition.

3.3 Identification with AdS_2-branes with Weak Electric Fields

It has been discussed in [10] that the physical D-brane (D-string) in AdS_3 background should be wrapped on the twined conjugacy class, which has the structure of AdS_2 space. Such an AdS_2-brane can be expressed as the next simple equation with respect to the global coordinates of AdS_3 (t, θ, ρ) (2.4);

$$\sinh \rho \sin \theta = \text{const.} \equiv \sinh \psi_0 \ , \tag{3.26}$$

where ψ_0 parameterizes the location of the AdS_2-brane and physically corresponds to the strength of the electric field on that brane. In fact, we can readily find from (3.26) that $\rho \geq \rho_{\text{min}} \equiv \psi_0$ and thus ψ_0 parametrizes the point nearest to the center of AdS_3-space, on the AdS_2-brane "bended" by the electric field.

We now try to identify the Cardy states (3.21) with the AdS_2-branes. Our arguments are summarized as follows:

1. By construction and evaluation of the cylinder amplitudes (3.24), it is clear that our Cardy states (3.21) can interact only with the short string sectors in both open and closed string channels.

   Moreover, the fact that we are now taking the discrete values of l implies that only the string modes propagating within the range $\rho \lesssim 1$ (in the unit of the AdS_3 scale l_{AdS}) can interact with the Cardy states (3.21). In fact, the wave function of the string states corresponding to l is known to behave as $\sim e^{-(l+1)\rho}$ with respect to the "radial coordinate" ρ ($l = -1$ corresponds to the Breitenlohner-Freedman bound [23]).

Therefore, since we are now working with $l = 0, 1, \ldots, k-2$, the corresponding wave functions exponentially damp at the length scale $\rho \sim 1$.

2. At least under the large k limit, our cylinder amplitudes must be interpreted as the summation of classical open string solutions. We will have non-trivial winding sectors generated by the spectral flows of the type; $t \rightarrow t + w\tau$, $\theta \rightarrow \theta + w\sigma$. Such classical solutions were discussed in the recent works [11, 13]. Especially, the explicit forms of classical open short string solutions connecting two AdS_2-branes labeled ψ_1, ψ_2 are given in [13] (up to the degrees of freedom of $SL(2;\mathbf{R})$ isometry);

$$\begin{cases} t = (\alpha + w)\tau \\ \theta = (\alpha + w)\sigma + \theta_0 \\ \rho = \rho_0 \ , \end{cases} \tag{3.27}$$

with the even winding $w \in 2\mathbf{Z}$, and the parameters α $(0 \leq \alpha \leq 1)$, θ_0 $(0 \leq \theta_0 \leq 2\pi)$, $\rho_0 (\geq 0)$ should satisfy

$$\begin{cases} \sinh \rho_0 \sin \theta_0 = \sinh \psi_1 \\ \sinh \rho_0 \sin(\alpha\pi + \theta_0) = \sinh \psi_2 \ . \end{cases} \tag{3.28}$$

For the odd winding $w \in 2\mathbf{Z} + 1$, we likewise have

$$\begin{cases} t = (1 - \alpha + w)\tau \\ \theta = (1 - \alpha + w)\sigma + \pi - \theta_0 \\ \rho = \rho_0 \ , \end{cases} \tag{3.29}$$

where $\alpha, \ \theta_0, \ \rho_0$ are the same as above.

On the other hand, the building blocks of open string amplitudes (3.24) are the characters $\chi_L^{SL(2)}(\tau)$ (3.25), which can be rewritten as

$$\chi_L^{SL(2)}(\tau) = \lim_{z \rightarrow 0} \left[- \sum_{w \in 2\mathbf{Z}} \frac{q^{-\frac{(L+1-kw)^2}{4k}} y^{\frac{L+1}{2} - \frac{kw}{2}}}{i\theta_1(\tau, z)} + \sum_{w \in 2\mathbf{Z}+1} \frac{q^{-\frac{(k-L-1-kw)^2}{4k}} y^{\frac{k-L-1}{2} - \frac{kw}{2}}}{i\theta_1(\tau, z)} \right] \ . \tag{3.30}$$

It is quite natural to interpret the zero-mode parts of this amplitudes as the summation over the classical solutions. In fact, one can easily find that the first term and the second term in (3.30) nicely correspond to the classical solutions (3.27) and (3.29) respectively, since the energy parameter α is quantized as $n/(k+2)$ because of the time-like compactification, and hence identified with $(L+1)/k$ under the large k limit. (The classical solution (3.27) possesses the classical conformal weight $-\frac{k+2}{4}\alpha^2 \approx -\frac{k}{4}\alpha^2$, which should correspond to the quantum value $-\frac{(L+1)^2}{4k} + \frac{1}{4k}$.)

We can also check the consistency of spectral flows in the open and closed string channels. The spectral flows which generate the classical solutions (3.27), (3.29) are equivalent to $U_w \otimes \tilde{U}_w$, (2.24) in the closed string channel, after exchanging the roles of τ and σ. These are in fact compatible with the gluing condition (2.17), as we noted before.

If we consider the dS_2-branes instead of the AdS_2-branes, the classical open string solutions with non-trivial winding numbers are generated by the spectral flows; $t \to t + w\sigma$, $\theta \to \theta + w\tau$, which are equivalent to (2.23) in the closed string channel and compatible with the gluing condition (2.13).

3. Recalling the experience of $SU(2)$ WZW model, it seems plausible to relate the labels of Cardy states L_1, L_2 with the parameters of brane positions ψ_1, ψ_2. However, we would immediately face to an apparent contradiction. In (3.24) the L-value appearing in the open string channel has upper and lower bounds. However, the corresponding parameter α in the classical solutions should not have such bounds depending on the brane positions ψ_1, ψ_2 as discussed in [13]. How should we fill this gap? Recall the fact that our Cardy states only includes the excitations of short strings propagating within the finite domain $\rho \lesssim 1$. This implies that the classical solutions (3.27), (3.29) we should compare with the open string spectrum have to satisfy the constraints $\rho_0 \lesssim 1$. We can hence expect the upper and lower bounds for α in (3.27), (3.29).

   Interestingly, assuming the identification $\psi_i \approx \dfrac{\pi}{2}\left(1 - \dfrac{2L_i}{k}\right)$ for sufficiently small $|\psi_i|$ (weak electric field) and requiring $\sinh \rho_0 \lesssim \sinh 1 \sim 1$, we can show by a simple geometrical consideration that

$$|L_1 - L_2| \lesssim k\alpha \lesssim \min(L_1 + L_2, 2k - L_1 - L_2)\ . \tag{3.31}$$

   This relation nicely reproduces the quantum truncation appearing in the cylinder amplitude (3.24). We thus claim that the quantum number L which labels the Cardy states should correspond to the parameter ψ_0 parametrizing the locations of AdS_2-brane. Our above observation supports this claim at least in the cases of $|\psi_0| << 1$.

Of course, since L can only take a value within the range $L = 0, 1, \ldots, k-2$, the number of allowed branes should be finite in our quantum analysis. (Especially, our Cardy states cannot describe the AdS_2-brane with the strong electric field of which whole world-volume locates outside the AdS radius.) This situation is quite similar as in the $SU(2)$ case, and has its origin in the existence of the unitarity bound.

We also point out that the $\mathbf{Z}_2$ symmetry $\psi_i \rightarrow -\psi_i$ corresponds to $L_i \rightarrow k-2-L_i$ and the cylinder amplitudes (3.24) possess this symmetry as is expected. The existence of quantum truncation is again essential to this fact.

3.4 Space-time Chiral Primaries in Open String Spectrum

Now, let us turn our attention to the superstring on the background $AdS_3 \times S^3 \times T^4$. Based on our analysis in the previous subsections and incorporating the $SU(2)$ WZW model and free fermions, we can determine the open string spectrum in this superstring theory. We denote the free fermions along the $SL(2;\mathbf{R})$ directions as ψ^3, $\psi^{\pm}$ and those of $SU(2)$ as χ^3, $\chi^{\pm}$.

We focus on the special class of physical states - "space-time (anti) chiral primary states", which are inevitably 1/2 BPS states. For the closed string sector such BPS states (or the corresponding vertex operators) are important in the context of AdS_3/CFT_2 correspondence and were investigated in [24, 25, 26, 6]. We will show that the BPS $AdS_2 \times S^2$-brane whose $SL(2;\mathbf{R})$ sector is described by our Cardy state (3.21) contains an infinite number of such excitations compatible with the spectral flows.

The Cardy state of the total system should have the structure

$$|L, L', \ldots\rangle_C \equiv |L\rangle_C^{SU(2)} \otimes |L'\rangle_C^{SL(2)} \otimes \cdots ,$$

where $|L\rangle_C^{SU(2)}$ is the Cardy state of the $SU(2)$ sector, $|L'\rangle_C^{SL(2)}$ denotes the Cardy state defined in (3.21), and $\cdots$ indicates the contributions from other sectors (free fermions and T^4 sector). Considering a specific cylinder amplitude with respect to such Cardy state, we typically obtain the amplitude such as

$$\sum_{L,L'} N_{L_1 L_2}^{L} N_{L_1' L_2'}^{L'} \chi_L^{SU(2)}(\tau) \chi_{L'}^{SL(2)}(\tau) \cdots ,$$

where $N_{L_1 L_2}^{L}$, $N_{L_1' L_2'}^{L'}$ denote again the fusion coefficients of $\widehat{SU}(2)_{k-2}$. It is the most important fact for our later discussion that the character $\chi_L^{SL(2)}(\tau)$ (3.25) contains the contributions from the infinitely many sectors with non-trivial winding numbers. This will lead us to a rich structure of the spectrum of BPS states.

Now, let us begin the analysis of on-shell BPS states. We shall only consider the NS sector here and work with the (-1)-picture. We also neglect the excitations along the T^4 direction for simplicity. First let us focus on the sector of primary states (in the usual sense of world-sheet) with $w = 0$, and consider the physical states including only one oscillator of free fermions ("level one states"). The on-shell condition imposes that the spin of $SL(2;\mathbf{R})$ sector should be equal to that of $SU(2)$. Namely, we must look for the BRST invariant states within the Hilbert space; $\hat{\mathcal{L}}_L \otimes \hat{\mathcal{D}}_L^+ (\otimes$ Hilbert space of the free fermions) or $\hat{\mathcal{L}}_L \otimes \hat{\mathcal{D}}_L^-$, where $\hat{\mathcal{L}}_L$ denotes the integrable representation with the spin $L/2$ of $\widehat{SU}(2)_{k-2}$. For example, in the Hilbert space $\hat{\mathcal{L}}_L \otimes \hat{\mathcal{D}}_L^+$, generic physical states (of level one) should have the form

$$\sum_{M,M',A} \alpha_{M,M',A} |L,M\rangle^{SU(2)} \otimes |L,M'\rangle^{SL(2)} \otimes \psi^A_{-1/2}|0\rangle_f \otimes ce^{-\phi}|0\rangle_{\rm gh} \\ + \sum_{M,M',a} \beta_{M,M',a} |L,M\rangle^{SU(2)} \otimes |L,M'\rangle^{SL(2)} \otimes \chi^a_{-1/2}|0\rangle_f \otimes ce^{-\phi}|0\rangle_{\rm gh} \,, \tag{3.32}$$

where $|L,M\rangle^{SU(2)}$ $(M = L, L-2, \ldots, -L)$ and $|L,M'\rangle^{SL(2)}$ $(M' = L+2, L+4, \ldots)$ denote the primary states (zero-mode states) belonging to $\hat{\mathcal{L}}_L$ and $\hat{\mathcal{D}}_L^+$ respectively. Such physical states are considered in [11] and it has been claimed that they correspond to the fluctuations around the classical solution which make the quadratic variation of DBI action vanish.

We shall now concentrate on the 1/2 BPS states as already advertised, which were studied in [24, 25, 26, 6] as the space-time (anti) chiral primaries in the closed string sector. Among the level one physical states, the chiral primaries are given by imposing the constraints $J^3_0 + K^3_0 = 0$ (J^A, K^a are the total $SL(2;\mathbf{R})$, $SU(2)$ currents including the fermionic contributions), and also the anti-chiral primaries are given by the constraints $J^3_0 - K^3_0 = 0$. Notice that $-J^3_0$ corresponds to the space-time energy operator (or the space-time conformal weight) and K^3_0 evaluates the space-time R-charge.

The list of chiral primaries is as follows[7];

$$|L,\psi^{\pm}\rangle^{(+)} \equiv |L,\pm L\rangle^{SU(2)} \otimes |L,\mp L\mp 2\rangle^{SL(2)} \otimes \psi^{\pm}_{-1/2}|0\rangle_f \otimes c_1 e^{-\phi}|0\rangle_{\rm gh}$$

[7]One can explicitly check that these states (3.33), (3.34) actually preserve a half of space-time SUSY, when using the space-time SUSY generators constructed in [3]. Moreover, with the help of Wakimoto free field representation, they are shown to be the (anti) chiral primary states with respect to the *full* space-time superconformal generators as discussed in [25].

$$|L,\chi^{\pm}\rangle^{(+)} \equiv |L,\pm L\rangle^{SU(2)} \otimes |L,\mp L\mp 2\rangle^{SL(2)} \otimes \chi^{\pm}_{-1/2}|0\rangle_f \otimes c_1 e^{-\phi}|0\rangle_{\rm gh} \ , \tag{3.33}$$

where $|L,L\rangle^{SU(2)}$ ($|L,-L\rangle^{SU(2)}$) denotes the primary state of highest (lowest) weight in the spin $L/2$ integrable representation of $\widehat{SU}(2)_{k-2}$, and similarly $|L,-L-2\rangle^{SL(2)}$ ($|L,L+2\rangle^{SL(2)}$) denotes the primary state of highest (lowest) weight in $\hat{\mathcal{D}}^-_L$ ($\hat{\mathcal{D}}^+_L$) of $\widehat{SL}(2;\mathbf{R})_{k+2}$.

The list of anti-chiral primaries is likewise given by

$$\begin{aligned} |L,\psi^{\pm}\rangle^{(-)} &\equiv |L,\mp L\rangle^{SU(2)} \otimes |L,\mp L\mp 2\rangle^{SL(2)} \otimes \psi^{\pm}_{-1/2}|0\rangle_f \otimes c_1 e^{-\phi}|0\rangle_{\rm gh} \\ |L,\chi^{\pm}\rangle^{(-)} &\equiv |L,\pm L\rangle^{SU(2)} \otimes |L,\pm L\pm 2\rangle^{SL(2)} \otimes \chi^{\pm}_{-1/2}|0\rangle_f \otimes c_1 e^{-\phi}|0\rangle_{\rm gh} \ . \end{aligned} \tag{3.34}$$

Next let us consider the sectors with the non-trivial winding number w. Generically, the spectral flows do not always map an on-shell state to another on-shell state, since they do not preserve the BRST charge. Fortunately, we can find that the on-shell (anti) chiral states nicely behave under the spectral flow. To observe this aspect we shall make use of the same idea as in [26].

Recall the spectral flow (2.21) in the $SL(2;\mathbf{R})$ sector;

$$\left\{ \begin{array}{l} U_w j^3_n U_w^{-1} = j^3_n + \dfrac{k+2}{2} w \delta_{n,0} \\ U_w j^{\pm}_n U_w^{-1} = j^{\pm}_{n\mp w} \ . \end{array} \right. \tag{3.35}$$

We extend the actions of spectral flow operator U_w to the other sectors as in [26];

$$\left\{ \begin{array}{l} U^{(+)}_w k^3_n U^{(+)\,-1}_w = k^3_n - \dfrac{k-2}{2} w \delta_{n,0} \\ U^{(+)}_w k^{\pm}_n U^{(+)\,-1}_w = k^{\pm}_{n\mp w} \end{array} \right. \tag{3.36}$$

$$\left\{ \begin{array}{l} U^{(+)}_w \psi^3_n U^{(+)\,-1}_w = \psi^3_n \\ U^{(+)}_w \psi^{\pm}_n U^{(+)\,-1}_w = \psi^{\pm}_{n\mp w} \end{array} \right. \tag{3.37}$$

$$\left\{ \begin{array}{l} U^{(+)}_w \chi^3_n U^{(+)\,-1}_w = \chi^3_n \\ U^{(+)}_w \chi^{\pm}_n U^{(+)\,-1}_w = \chi^{\pm}_{n\mp w} \ . \end{array} \right. \tag{3.38}$$

We also define

$$\left\{ \begin{array}{l} U^{(-)}_w k^3_n U^{(-)\,-1}_w = k^3_n + \dfrac{k-2}{2} w \delta_{n,0} \\ U^{(-)}_w k^{\pm}_n U^{(-)\,-1}_w = k^{\pm}_{n\pm w} \end{array} \right. \tag{3.39}$$

$$\left\{ \begin{array}{l} U^{(-)}_w \chi^3_n U^{(-)\,-1}_w = \chi^3_n \\ U^{(-)}_w \chi^{\pm}_n U^{(-)\,-1}_w = \chi^{\pm}_{n\pm w} \ , \end{array} \right. \tag{3.40}$$

and the action of $U_w^{(-)}$ on the ψ sector is defined to be the same as $U_w^{(+)}$.

For the $SU(2)$ sector, (3.36), (3.39) mean nothing but the actions of affine Weyl group and we can observe

$$\begin{array}{llll} U_w^{(\pm)} : \hat{\mathcal{L}}_L & \longrightarrow & \hat{\mathcal{L}}_L & (w \in 2\mathbf{Z}) \\ U_w^{(\pm)} : \hat{\mathcal{L}}_L & \longrightarrow & \hat{\mathcal{L}}_{k-2-L} & (w \in 2\mathbf{Z}+1) \ . \end{array} \tag{3.41}$$

Now, the important thing is as follows: as discussed in [26], the spectral flow operators $U_w^{(\pm)}$ have the properties;

$$\begin{aligned} U_w^{(\pm)} Q_{BRST} U_w^{(\pm)\,-1} &= Q_{BRST} - w \Big[c(0)(J^3 \pm K^3)(0) \\ &\qquad + \sqrt{\frac{k}{2}} \eta(0) e^{\phi(0)} (\psi^3 \pm \chi^3)(0) \Big] \ , \end{aligned} \tag{3.42}$$

$$U_w^{(\pm)} (J_0^3 \pm K_0^3) U_w^{(\pm)\,-1} = J_0^3 \pm K_0^3 \ . \tag{3.43}$$

They imply that *the actions of $U_w^{(+)}$ ($U_w^{(-)}$) close in the space of the on-shell space-time (anti) chiral primary states.* We also point out the next identities

$$\begin{array}{ll} U_{-1}^{(+)} |L, \psi^+\rangle^{(+)} = |k-2-L, \chi^-\rangle^{(+)} \ , & U_{-1}^{(+)} |L, \chi^+\rangle^{(+)} = |k-2-L, \psi^-\rangle^{(+)} \ , \\ U_{-1}^{(-)} |L, \psi^+\rangle^{(-)} = |k-2-L, \chi^+\rangle^{(-)} \ , & U_{-1}^{(-)} |L, \chi^-\rangle^{(-)} = |k-2-L, \psi^-\rangle^{(-)} \ . \end{array} \tag{3.44}$$

Using these identities, the complete set of the on-shell chiral primaries is given by

$$\{ \, |L, w, \psi^+\rangle^{(+)} \equiv U_w^{(+)} |L, \psi^+\rangle^{(+)}, \ |L, w, \chi^+\rangle^{(+)} \equiv U_w^{(+)} |L, \chi^+\rangle^{(+)} \ \ (w \in \mathbf{Z}, \ L = 0, 1, \ldots, k-2) \, \}, \tag{3.45}$$

and for the anti-chiral primaries, we likewise obtain

$$\{ \, |L, w, \psi^+\rangle^{(-)} \equiv U_w^{(-)} |L, \psi^+\rangle^{(-)} \ , \ |L, w, \chi^-\rangle^{(-)} \equiv U_w^{(-)} |L, \chi^-\rangle^{(-)} \ \ (w \in \mathbf{Z}, \ L = 0, 1, \ldots, k-2) \, \} \ . \tag{3.46}$$

To summarize, we have obtained an infinite number of on-shell (anti) chiral primaries (3.45) ((3.46)). The L-value suffers the quantum truncation depending on the choice of Cardy states (of both $SU(2)$ and $SL(2;\mathbf{R})$ sectors). On the other hand, the spectral flows can act on this spectrum transitively for arbitrary even windings w irrespective of the choice of Cardy states. For the odd windings, it is generically not the case, since they act as the $\mathbf{Z}_2$-reflection $L \to k-2-L$ both on the $\widehat{SU}(2)_{k-2}$ and $\widehat{SL}(2;\mathbf{R})_{k+2}$ sectors. The related arguments based on the analysis of classical solution are given in [11, 13].

We would like to point out that the spectrum of closed string channel (Cardy states) also includes an infinite number of such chiral primary states with arbitrary winding numbers, which can be constructed in the similar manner as those of the open string spectrum (3.45), (3.46). The essential point is that the spectral flows of the type (2.24) preserve the Cardy state of $SL(2;\mathbf{R})$ sector (3.21) (up to the signature)

$$U_w \otimes \tilde{U}_w |L\rangle_C^{SL(2)} = (-1)^{Lw} |L\rangle_C^{SL(2)} \ , \tag{3.47}$$

by our construction. The similar relations hold also for the other sectors (the $SU(2)$ sector and the sectors of free fermions).

3.5 Comments on Boundary States Based on Continuous Series

Our above discussions may be incomplete, since the considerations about the long strings are lacking in both the open and closed string channels. A priori, the correct boundary states describing the AdS_2-branes are expected to have the contributions also from the continuous series. The piece of boundary states composed of the discrete series, which we constructed, describes the open short strings propagating in the domain bounded by the AdS radius, and all the space-time chiral primary states (as both open and closed strings) appear in this sector.

It is an easy task to construct the Ishibashi states for the continuous series. However, we will have a large ambiguity when we intend to make the Cardy states from them. We could expect the following open string excitations from such Cardy states based on the continuous series;

(1) the open long strings with arbitrary winding number w.

(2) the open short strings with arbitrary winding number w which propagates in the domain $\rho \gtrsim 1$ (AdS length).

The classical solutions of open long strings with arbitrary winding numbers are explicitly constructed in [13] and it is natural to expect the corresponding excitations (1) in the quantum open string spectrum. It can be easily shown by the consideration of modular weight that we must take the boundary states based on the continuous series with *no winding* in order to obtain the summation over windings in the open string channel. Conversely, if we start with the boundary states with the summation over windings, we will obtain the open string spectrum with no winding. Therefore, it seems difficult to naively construct the Cardy states so as to be consistent with the spectral flow symmetry in *both* open and closed string channels in this case. This is one of the main puzzles and we need further detailed investigations about this problem.

To assume the second excitations (2) is also quite natural, since the Cardy states considered here are expected to interact with the strings which can propagate in the region far away from the center, in contrast to those we discussed in the previous subsections. In fact, choosing some hyperbolic functions as the wave functions,

$$\Psi_{\Lambda}(\lambda) \sim \sinh\left(\frac{\pi\Lambda\lambda}{k}\right) , \tag{3.48}$$

we can obtain the open string spectrum of discrete series as presented in [12]. However, at least under a naive consideration, we would face to the following difficulties, if we start with the same criterion that we took for the previous arguments;

1. It seems difficult to incorporate the non-trivial winding sectors in the open string channel.

2. It seems difficult to make the open string spectrum compatible with the unitarity bound.

From these reasons, it is unclear so far whether we can reproduce the consistent open string spectrum from the boundary states based on the continuous series. It is an important problem which should be resolved in the future studies[8].

4 Discussions

In this paper we have studied the AdS_2-branes in the string theory on AdS_3 background from the viewpoints of boundary states, emphasizing the role of open-closed duality in string theory. We have constructed the Cardy states based on the discrete series, which possess the following good properties;

1. They are compatible with the symmetry of spectral flow.

2. They are consistent with the unitarity and normalizability in the open string spectrum.

[8]Recently, this problem is discussed in [27].

We have observed that the first property brings us a rich structure of physical BPS states both in the open and closed string channels, that is, the spectral flows consistently act on the spectrum of infinitely many space-time chiral primaries. Such physical BPS states are expected to play important roles in the context of AdS_3/CFT_2 correspondence. One interesting direction for future works is the attempt to describe the D-branes in AdS_3 string theory in the framework of the boundary CFT (in the sense of AdS/CFT correspondence, though this terminology is sometimes very confusing!) rather than the world-sheet CFT approach. The analysis of physical BPS states given in this paper will bring us helpful insights for the studies along such direction.

The second property is originating from the quantum truncation like the fusion rule in the $SU(2)$ WZW model. The existence of such truncation seems to lead us to the "fuzziness" of brane dynamics as in the $SU(2)$ case. One might feel such fuzziness peculiar, since AdS_3 and AdS_2 are non-compact spaces in contrast to $SU(2)$ and S^2. In fact, the classical analysis given in [13] suggests that we should not have such truncation in the open string spectrum. One possibility to resolve this contradiction may be to claim that our Cardy states (3.21) should *define* the "fuzzy AdS_2-branes" which *do not have the counterparts in the classical brane geometry.* More modestly speaking, (3.21) should correspond to the piece of Cardy state composed of discrete series and we will have to further take account of the piece composed of continuous series suitably in order to get the complete Cardy states describing the "classical" AdS_2-branes.

As we discussed, the Cardy states (3.21) can interact only with the short strings propagating in the domain bounded by the AdS radius. This fact is supposed to be the origin of fuzziness mentioned above. On the other hand, it is natural to expect that the piece composed of continuous series well describes (1) the open long strings which can reach the asymptotic region (the boundary of AdS_3), and (2) the open short strings propagating in "far region from the center". If this is indeed the case, we will be able to obtain the boundary states reproducing the classical geometry of AdS_2-branes. However, as we mentioned in the last part of section 3, we have several puzzles about the Cardy states based on the continuous series so far, and they have to be clarified in future studies.

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Appendix A Notations

The theta functions are defined as follows. Set $q \equiv e^{2\pi i\tau}$ and $y \equiv e^{2\pi iz}$;

$$\begin{aligned}
\theta_1(\tau,z) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\sin(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^m),\\
\theta_2(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\cos(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^m),\\
\theta_3(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2}y^{n} \equiv \prod_{m=1}^{\infty}(1-q^m)(1+yq^{m-1/2})(1+y^{-1}q^{m-1/2}),\\
\theta_4(\tau,z) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}y^{n} \equiv \prod_{m=1}^{\infty}(1-q^m)(1-yq^{m-1/2})(1-y^{-1}q^{m-1/2})\ ,
\end{aligned} \tag{A.1}$$

and also,

$$\Theta_{m,k}(\tau,z) = \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2}y^{k(n+\frac{m}{2k})}. \tag{A.2}$$

We also set

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\ , \tag{A.3}$$

and often use the identity

$$\partial_z\theta_1(\tau,z)|_{z=0} = 2\pi\eta(\tau)^3\ . \tag{A.4}$$

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